The dynamics of spheroidal masses of buoyant fluid

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(Received 17 February 1964)

It is shown how a simple property of the spherical vortex model can be used to investigate the dynamics of a buoyant, expanding thermal. No details of the vortex motion are required, only the fact that the flow round the buoyant region is potential. The main result is a demonstration that there is a relation between the two constants C and α arising in the dimensional analysis (where in the usual notation $w = C(\Delta r)^{\frac{1}{2}}$ and $r = \alpha z$), which have up till now been measured separately and treated as independent. The analysis has been extended to spheroidal thermals by calculating the virtual mass for the appropriate outline, and it has also been generalized to include thermals in which the total buoyancy is increasing with time.

Using these results, and an earlier experimental verification that the mean angles of spread are nearly the same under various conditions of stability, *it is suggested that the whole of the mean behaviour of a thermal can be calculated in two nearly independent steps.* First, the density difference or Δ as a function of height may be calculated using purely kinematic equations of conservation. Secondly, the velocity is obtained from the local values of Δ and radius r, using a mean value of C, since this has now also been shown to vary little over a wide range of conditions.

1. Introduction

The motion of buoyant 'thermals' in surroundings at rest has been closely studied in recent years, both theoretically and experimentally. The basic results of dimensional analysis have been supported by laboratory experiment, from which numerical values of various constants have been obtained, and by more detailed numerical calculations. The velocity w of the top and the radius r of a thermal may be written

$$w = C(\Delta r)^{\frac{1}{2}}, \quad r = \alpha z = z/n, \tag{1}$$

where z is the height of the front of the thermal, $\Delta = g(\rho_1 - \rho)/\rho_1$, ρ and ρ_1 are the densities inside and outside the thermal, and g is the acceleration due to gravity. For the case of a thermal moving through uniform surroundings Scorer (1957) showed experimentally that in any given experiment C and α remain constant, there being, however, considerable variation from one experiment to another. His mean values are $C \doteq 1{\cdot}2$ and $\alpha = 1/n = \frac{1}{4}$. Turner (1963) extended the experiments to several cases of increasing total buoyancy, and showed that α is again constant with a not very different mean value, though he was not able to evaluate C from these experiments.

It has usually been assumed that the constants C and α must be evaluated *independently* using the appropriate measurements. Richards (1961) has, it is true, developed the vortex ring model of Turner (1957) to derive a functional form for the relation between the angle of spread and the rate of progress of thermals, but he regards the numerical factor as empirical and obtainable only from both sets of measurements. Richards (1963) later showed, however, that the factor so derived is consistent with the numerical integration due to Ogura (1962), which suggests that one might seek a more fundamental explanation for the relation between the two quantities.

It is the main purpose of this paper to show how ideas obtained from the model of a spherical, or better, spheroidal, vortex may be used to investigate the dynamics of a thermal, and in particular to derive a numerical relation between C and α. To do this it is not necessary to specify the details of the interior motion, as was done by Levine (1959), but only certain overall properties of the exterior flow.

2. The spherical vortex model

Woodward (1959) has shown experimentally that the motion inside a thermal is instantaneously rather like that in a slightly flattened spherical vortex. As the size increases, the velocity distribution remains similar at all times with a sharp boundary between turbulent, buoyant fluid and the environment. Levine (1959) went further and used Hill's model of a spherical vortex (which is described in Lamb 1932), but he considered the case of constant size and a turbulent interchange in both directions across its edge. Recently Turner (1964) has pursued the consequences of assuming an *expanding* Hill's spherical vortex, and has shown that most of the kinematic features of Woodward's experiments can be described well in this way.

The basic fact about this last model which we shall need to use is that the motion at any instant outside a spherical boundary of increasing radius is just the same as the potential flow round a solid sphere of the same size. The circulating motion inside is also specified in the model, but since only the linear momentum equation is required here, only the motion of the centre of mass is relevant. Although the flow round the vortex can be regarded as frictionless, 'resistance to motion' arises in two ways; by the incorporation of external fluid into the expanding sphere, and as a virtual mass effect due to the displacement of fluid round the thermal. It will later be seen how these two terms can be evaluated for shapes other than spherical.

It was shown by Turner (1957) how extra information about the motion of a thermal can be obtained by regarding it as a special case of a buoyant vortex ring, and the result we are seeking now represents an extension of this earlier work. The impulse associated with a system of circular vortices is of the form

$$P = \pi \rho K R^2, \tag{2}$$

where K is the circulation round the ring, and R is the radius of the circular axis of the system of vortices (Lamb 1932, p. 239). In a uniform environment the assumption of constant total buoyancy combined with that of similarity implies that K is constant; its value is determined by the conditions of formation of the vortex. For Hill's spherical vortex the result (2) may be put in a form which no longer involves K explicitly, but only the velocity w_0 of the centre of the sphere and r its radius, namely

$$P = 2\pi\rho r^3 w_0, \tag{3}$$

$$= \tfrac{3}{2}\rho V w_0, \tag{3a}$$

where V is the volume of the sphere. The factor $\frac{3}{2}$ is just the virtual mass coefficient by which the impulse of the sphere regarded as a solid body must be increased to give that of all the fluid set into motion. This is a verification in one particular case of a more general result; equation (3a) could in fact have been written down immediately without reference to the interior motion.

The buoyancy force acting to increase the impulse of the system is

$$\rho F = \rho\Delta V, \tag{4}$$

and we can write an equation for the rate of change of the impulse

$$dP/dt = \rho F. \tag{5}$$

Let us define C_0 and α_0 to be the corresponding constants to those in (1) if w_0 and z_0, the velocity and height of the centre, are used instead of those at the top. Then using (3), and remembering that $K \propto rw_0$ remains constant, it follows that

$$dP/dt = 4\pi\rho r^2\alpha_0 w_0^2 = \rho\Delta V.$$

This reduces to

$$w_0^2 = r\Delta/3\alpha_0, \tag{6}$$

or, comparing with (1),

$$C_0^2\alpha_0 = \tfrac{1}{3}. \tag{7}$$

Thus in terms of the properties of the *centre* of a spherical vortex the desired result has been obtained, i.e. a numerical relation between C_0 and α_0.

This can be immediately converted into a relation between C and α by using an identity obtained from the geometry of the expanding sphere. By definition $z = z_0 + r$, so that $dz_0/dt = w_0 = w(1-\alpha)$. Elimination of Δ and r from (1) and the similar expressions for the centre of the sphere gives

$$wC_0 = w_0 C \quad \text{and} \quad w\alpha = w_0\alpha_0.$$

Combining these last three equalities we obtain

$$C_0^2\alpha_0 = C^2\alpha(1-\alpha) = \tfrac{1}{3}. \tag{8}$$

If the experimental mean value of $\alpha = \frac{1}{4}$ is substituted directly, we obtain $C = \frac{4}{3}$ which is rather higher than that observed for C. In the next section, however, we shall show that closer agreement is obtained if proper allowance is made for the observed flattening of thermals.

3. Spheroidal boundaries

The idea on which the above calculation is based may be applied more generally. For all shapes of thermals, there will be a boundary, close to the visible edge, outside which frictional effects are negligible, and the flow therefore potential. The change in shape affects the dynamics through its effect on the 'virtual mass' of the solid body of the same shape, which depends only on this exterior flow. For definiteness and ease of calculation, it is assumed here that the outline can be approximated by spheroids of various eccentricities.

The required results are readily available, and are given in a convenient form by Ramsey (1935, p. 186). The virtual mass coefficient C_v, to replace the factor $\frac{3}{2}$ in (3a), is for an oblate spheroid of revolution

$$C_v = 1+\frac{\tan\theta-\theta}{\theta-\sin\theta\cos\theta}, \tag{9}$$

where $e = \sin\theta$ is the eccentricity of the elliptical section. The values calculated for various ratios r/b of major to minor axis of the ellipse are shown in table 1, together with the shape factor

$$m = V/r^3 = \tfrac{4}{3}\pi(b/r).$$

It is interesting to note that over this range of r/b, 1·0–2·0, the tabulated C_v are within 5 % of that obtained by assuming that the addition to the impulse due to the flow round the spheroid is the same as for a sphere of radius r, i.e. $C_v \approx 1+(r/2b)$, This value is also shown in table 1.

Following through the arguments leading to (7) and (8), it is found that these must now be replaced by

$$C_0^2\alpha_0 = (2C_v)^{-1}$$

$$\approx b/(2b+r), \tag{10}$$

and

$$C_0^2\alpha_0 = C^2\alpha(1-\alpha b/r). \tag{11}$$

With the values $\alpha = \frac{1}{4}$ and $m = 3$ suggested by Scorer (1957), we obtain $C_0^2\alpha_0 = 0{\cdot}29$ (instead of $\frac{1}{3}$ for the spherical vortex) and $C = 1{\cdot}2$, in close agreement with the mean experimental value. The use of a smaller value of m, as suggested by the results of Saunders (1962), leads to a slightly lower value of C, but still within the experimental uncertainty.

It will now be shown how the same results may be used to make a numerical estimate of the factor determined empirically by Richards (1961). Using our notation, he showed that if

$$z^2 = k_1 t, \tag{12}$$

then his results could be represented by

$$k_1 = C_1 n^{\frac{3}{2}}(V\Delta)^{\frac{1}{2}}, \tag{13}$$

where $n = 1/\alpha$ as defined in (1) and C_1 is about 0·73. The following analysis will show that C_1 is not strictly constant, but depends only weakly on the shape and angle of spread.

Using (1) and (12) it follows that

$$k_1 = 2(r/\alpha)\,C(r\Delta)^{\frac{1}{2}}. \tag{14}$$

Eliminating C using the results (10) and (11) appropriate to a spheroid, and grouping the terms to bring the expression into the form (13), gives

$$k_1 = n^{\frac{3}{2}}\frac{3\sqrt{2}\,r}{4\pi\,b}\left(\frac{m}{C_v}\right)^{\frac{1}{2}}\left(1-\frac{b}{r}\alpha\right)^{-\frac{1}{2}}(V\Delta)^{\frac{1}{2}}, \tag{15}$$

or

$$C_1^2 = \frac{3}{2\pi}\frac{r}{b}\frac{1}{C_v}\left(1-\frac{b}{r}\alpha\right)^{-1}. \tag{16}$$

Notice that the last term in this expression arises only because properties at the top of thermals are usually measured; if everything were referred to the centre of gravity it would be absent and the whole dependence of k_1 on n would be like $n^{\frac{3}{2}}$. Although there are several variable factors in (16) (it is strictly constant only for a fixed shape and angle of spread), their combined effect causes only small changes in C_1. The variation of the term $(r/b)^{\frac{1}{2}}C_v^{-\frac{1}{2}}$ is shown, for example, in the last column of table 1. The dependence on α is also small, and the value corresponding to $\alpha = \frac{1}{4}$ and $m = 3$ is $C_1 = 0{\cdot}69$, in good agreement with Richards' estimate, $C_1 = 0{\cdot}73$.

r/b	$e = \sin\theta$	m	C_v	$1+\dfrac{r}{2b}$	$\left(\dfrac{r}{b}\dfrac{1}{C_v}\right)^{\frac{1}{2}}$
1·0	0	4·19	1·50	1·50	0·82
1·2	0·552	3·49	1·62	1·60	0·86
1·4	0·700	3·00	1·74	1·70	0·90
1·6	0·780	2·62	1·87	1·80	0·93
1·8	0·830	2·33	1·99	1·90	0·95
2·0	0·865	2·10	2·10	2·00	0·98

TABLE 1. The properties of spheroidal bodies having various ratios r/b of major to minor axis and eccentricity e. m is the shape factor defined by $V = mr^3$, and C_v the virtual mass coefficient as defined in the text.

As previously stated, this result also sheds light on the close agreement of Ogura's model with some of the experimental results. The existence of a particular value of n in his calculations should probably be regarded as fortuitous, and dependent on the initial distribution of buoyancy and the way this generates circulation during the acceleration from rest. Given a certain value of n, however, we have just seen that the dynamics of a buoyant region, in particular the quantity k_1, depends mainly on the total buoyancy through the factor $(V\Delta)^{\frac{1}{2}}$ and the angle of spread through $n^{\frac{3}{2}}$. The dependence on the shape of the region, and the virtual mass coefficient, is slight. Now the motions calculated by Ogura may also be regarded as equivalent, instantaneously, to the translation of a solid body and the potential flow around it, plus an interior flow with no net impulse. For any shape not too different from spherical the arguments used above would again apply, so that we would expect the value of the factor C_1 to differ little from those just deduced. This is indeed borne out by the numerical results.

4. Thermals with increasing total buoyancy

Similar arguments may be used to study the dynamics of thermals whose total buoyancy is changing in a specified manner. Turner (1963) has recently carried out experiments which show that thermals still spread linearly, with a slightly smaller angle, when the acceleration or the velocity is constant, but through uncertainties in the density measurements little reliable quantitative information was obtained about the dynamics. It will now be shown how this gap may be filled theoretically.

(a) Constant acceleration

As shown in the paper just mentioned, constant acceleration a of the top implies that the mean density difference between the thermal and its surroundings is constant, or

$$a = \beta\Delta, \tag{17}$$

where β is a constant. Although the same equation (5) may be written down to describe the motion at any time, in evaluating the derivative we must now keep in mind the fact that Δ and not rw_0 remains constant. Thus we obtain for a spherical vortex instead of (8):

$$C_0^2\alpha_0 = C^2\alpha(1-\alpha) = \tfrac{4}{21}, \tag{18}$$

i.e. still the same form of relation between C and α but with a different constant. For a flattened thermal (10) must be modified similarly, and the term $\frac{1}{2}C_v^{-1}$ replaced by $\frac{2}{7}C_v^{-1}$. Experimentally it was observed that the mean m was 4·3, so that these thermals were nearly spherical, and the mean α was 0·20. The corresponding value of C is 1·1.

It is also possible to obtain a theoretical value for β to compare with the experimental upper limit, which was $\beta^* = 0{\cdot}34$. The changing buoyancy in these experiments was produced by the chemical release of small bubbles, and the total volume of gas and hence Δ estimated by measuring the volume of water displaced by the gas and overflowing into a side tube. Because the buoyancy was changing so rapidly, it was suspected at the time that this measurement underestimated Δ and therefore overestimated β.

Using (1), and remembering that Δ is constant, we obtain for the case of the sphere,

$$\begin{aligned} a &= dw/dt = \tfrac{1}{2}C\Delta^{\frac{1}{2}}r^{-\frac{1}{2}}dr/dt \\ &= \tfrac{1}{2}C^2\alpha\Delta \\ &= \tfrac{2}{21}[\Delta/(1-\alpha)] \quad \text{from (18).} \end{aligned} \tag{19}$$

Thus with $\alpha = 0{\cdot}20$, $\beta = 0{\cdot}12$, or about one-third of β^*. The experimental value is indeed a serious overestimate.

(b) Constant velocity

If the total buoyancy within the thermal is changing with time in such a way that the mean velocity w remains constant, the derivative dP/dt in the impulse equation may again be evaluated. In this case we find instead of (8):

$$C_0^2\alpha_0 = C^2\alpha(1-\alpha) = \tfrac{2}{9}. \tag{20}$$

For the spheroidal case, the right-hand side of (10) will become $(3C_v)^{-1}$. With the experimental mean values of $m = 2{\cdot}9$ and $\alpha = 0{\cdot}23$ determined for thermals having constant velocity, $C = 1{\cdot}0$.

5. The application to the atmosphere

The results obtained in the previous section, together with the laboratory experiments referred to there, have important implications which have been mentioned briefly elsewhere (Turner 1963), but merit a more detailed discussion. The numerical values have, for ease of reference, been collected together in table 2.

The application of the laboratory results which we have had in mind is of course the growth of cloudy thermals in the atmosphere. Before attempting to make this comparison, we should consider again how closely the laboratory motion, of an element with changing buoyancy in neutral surroundings, can be said to model the more usual case in the atmosphere where an initally statically stable environment is made unstable by the release of extra buoyancy in the form of latent heat. In both cases the instability can be confined to the moving region, and this being so we might expect that the behaviour would be similar in the two cases

Type of thermal	α	m	C
Neutral surroundings	0·25	3·0	1·2
Constant velocity	0·23	2·9	1·0
Constant acceleration	0·20	4·3	1·1
Typical values	0·23	—	1·1

TABLE 2. Summarizing the properties of thermals in various conditions of stability. The half angle of spread α and the shape factor m are mean values taken from the laboratory experiments reported previously, and C has been deduced from the theory of the present paper.

provided the mean density *differences* between the thermal and its surroundings have the same time history. There are two important qualifications: the density gradients in the environment should not be so large that erosion of the element occurs, and no substantial energy should be carried away in the form of waves in the stratified medium, thus introducing a term into the drag which is not included in (5). The results of Warren (1960) suggest that the wave energy is negligible unless the thermal is about to come to rest because its buoyancy has reversed; and erosion is also only important in these circumstances. For all the cases considered in this paper, for which the total buoyancy is either constant or increasing, the comparison between laboratory experiments and the atmosphere should be valid.

The most important fact about the results in table 2 is their relative insensitivity to the stability of the thermals: whether they are accelerating or decelerating, the spread has been shown to be linear, with only a small range of mean angles, and the relation between the velocity and the local properties is always nearly the same. This implies that the variation of internal properties with time, and the dynamics of the elements, can be treated as separate problems as follows.

(*a*) *Implications of the linear spread of thermals*

First, with a linear spread, the fractional rate of entrainment per unit height E (originally defined by Stommel 1947) assumes an especially simple form:

$$E = \frac{1}{V}\frac{dV}{dz} = \frac{1}{r^3}\frac{dr^3}{dz} = \frac{3\alpha}{r} = \frac{3}{z}, \tag{21}$$

where z is the distance from the virtual origin. This inverse dependence of rate of entrainment on element size has of course been suggested before, but we are now able to put closer limits on the values of the constants. It is possible in fact to use a single mean value of α, say $\alpha = 0{\cdot}23$, to represent the entrainment in all unstable situations. The variations of α will be much less important than those due to the large range of sizes encountered. As examples we might take $r = 3$ km for which $E \approx 0{\cdot}23\,\text{km}^{-1}$, and $r = \frac{1}{2}$ km, $E \approx 1{\cdot}4\,\text{km}^{-1}$; these are consistent with experimental values reported in the literature for thunderstorm towers and trade wind cumuli respectively.

Now the dimensional arguments leading to a linear spread of thermal are entirely equivalent to the entrainment assumption, that fluid is entering the surface of the element at a mean rate which is proportional to the upward velocity. Thus for a spherical element the equation

$$\frac{d}{dt}\left(\tfrac{4}{3}\pi r^3\right) = 4\pi r^2 \alpha w \qquad (22)$$

implies $r = \alpha z$, and the entrainment constant is just the tangent of the half angle of spread. (For non-spherical elements there will of course be a factor introduced, due to the change in the surface to volume ratio, but the numerical differences are small and for simplicity only the spherical case will be considered here.) This result is purely kinematic, and independent of the properties of the environment and the dynamics of the motion. Again we may use a single typical value of α in all cases.

(b) Conservation relations for other properties

The linear spread with height allows one to predict not only the rate of entrainment of mass from the environment, but also the variation of any property associated with the entrained fluid. Consider the variation within a thermal of any property which is conserved on mixing and which has a concentration say $q(z)$ inside the element and $q_1(z)$ in the environment. The equations will again be developed for an incompressible fluid, but the extensions to the atmosphere where extra expansion is caused by pressure changes is immediate if concentrations per unit mass are used.

The essential point is that these calculations use only *conservation* equations for the property under consideration, once the change in size has been specified as a function of height. Thus for a spherical element an equation for q may be derived from (22) in the form

$$d(\tfrac{1}{3}r^3 q)/dt = \alpha r^2 w q_1. \qquad (23)$$

Combining (22) and (23) gives

$$r\,dq/dt = 3\alpha w(q_1 - q),$$

or

$$z\,dq/dz = 3(q_1 - q). \qquad (24)$$

The derivation of (24) has not used any specific assumption about the dependence of q_1 on height. It may easily be solved, for example, for an element starting from a point or a finite source with a power-law concentration gradient. As a simple example consider a linear variation and a point source, i.e. put $q_1 = q_0 + \epsilon z$. In this case dq/dz is also constant and from (24) it must be equal to $\frac{3}{4}\epsilon$. Thus the *difference* in concentration of the property q between the inside and the outside in a unit height interval is $\frac{1}{4}\epsilon$, or one quarter of the value it would have if no mixing took place. This result (which is independent of α provided the ratio of final to initial radius is large) was obtained by Ludlam (1958) in another form. The angle of spread does of course enter into solutions of (24) which begin with a finite size, but the 'memory' of initial properties is appreciable only over displacements comparable with the diameter of the element. Warner (1963) has recently discussed this case of finite initial radius.

Let us now suppose that q refers first to the total moisture, and then to the potential temperature of the element assuming no condensation takes place. The variation of the mean internal values of these two quantities with height may be found in simple cases by exact, or in more complex situations by numerical, integration of (24) when $q_1(z)$ is known. Condensation can then be considered, using the known dependence of saturation mixing ratio on the temperature, and the final virtual temperature difference and liquid water concentration calculated.

Thus the experimental result that there is a linear spread with only small variations of mean α over a wide range of stability conditions, leads finally to estimates of r and Δ (or the density difference between a thermal and its surroundings) at any height. These estimates have been made without saying anything about the dynamical behaviour and this may now be considered as a separate, nearly independent step.

(c) *Estimates of thermal velocity*

The calculations of section (4) have shown that for a wide range of stabilities, neutral to very unstable, the motion of a thermal can be well described by equations of the form (1) with only small variations of the numerical factor C from case to case. That is, the velocity is determined mainly by the local Δ and r, and only to a minor extent by the fact that the thermal is accelerating or decelerating. There will still be variations of n between individual experiments which cannot be predicted in detail, but if the mean behaviour is of interest, we see from table 2 that $C = 1{\cdot}1$ is the appropriate value to use in (1). The quantities Δ and r might of course be available directly from experiment, but if not, the previous section has shown how mean values can in principle be calculated from the properties of the *environment*.

It should be emphasized again that one important exception must be made in the application of either the kinematic or dynamic result. Neither will be valid in very stable environments, when a thermal is losing buoyancy rapidly and overshooting the level of zero buoyancy. First, the assumption of linear spread may break down there, as discussed by Turner (1960). Secondly, the phenomena of erosion and wave generation will make a simple impulse equation like (5) no longer relevant, since it leaves out important contributions to the drag. In all cases where cloud thermals are growing vigorously in a still environment, however, both parts of the calculation should certainly be applicable.

This work was carried out while the author was on leave from the C.S.I.R.O. Division of Radiophysics, Sydney, Australia. I am indebted to the Woods Hole Oceanographic Institution for the award of the Rossby Fellowship, and to Dr Claes Rooth for his helpful criticisms. This is contribution No. 1469 of the Woods Hole Oceanographic Institution, and was supported in part by NSF Contract No. GP 317.

REFERENCES

LAMB, H. 1932 *Hydrodynamics*. Cambridge University Press.

LEVINE, J. 1959 *J. Meteor.* **16**, 653.

LUDLAM, F. H. 1958 *Nubila*, **1**, 12.

OGURA, T. 1962 *J. Atmos. Sci.* **19**, 492.

RAMSEY, A. S. 1935 *A Treatise on Hydromechanics*. London: Bell.

RICHARDS, J. M. 1961 *J. Fluid Mech.* **11**, 369.

RICHARDS, J. M. 1963 *J. Atmos. Sci.* **20**, 241.

SAUNDERS, P. M. 1962 *Tellus*, **14**, 177.

SCORER, R. S. 1957 *J. Fluid Mech.* **2**, 583.

STOMMEL, H. 1947 *J. Meteor.* **4**, 91.

TURNER, J. S. 1957 *Proc. Roy. Soc.* A, **239**, 61.

TURNER, J. S. 1960 *J. Fluid Mech.* **7**, 419.

TURNER, J. S. 1963 *Quart. J. R. Met. Soc.* **89**, 62.

TURNER, J. S. 1964 *J. Fluid Mech.* **18**, 195.

WARNER, J. 1963 *J. Atmos. Sci.* **20**, 546.

WARREN, F. W. G. 1960 *J. Fluid Mech.* **7**, 209.

WOODWARD, B. 1959 *Quart. J. R. Met. Soc.* **85**, 144.